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Hyperelliptic solutions of modified Korteweg–de Vries equation of genus g: essentials of the Miura transformation

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Abstract

Explicit hyperelliptic solutions of the modified Korteweg–de Vries equations without any ambiguous parameters were constructed in terms of only the hyperelliptic al-functions over the non-degenerate hyperelliptic curve $y^2 = f(x)$ of arbitrary genus g. In the derivation, any θ-functions or Baker–Akhiezer functions were not essentially used. Then the Miura transformation naturally appears as the connection between the hyperelliptic $\wp$-functions and hyperelliptic al-functions.

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1. Introduction

It might not be an exaggeration to say that the discovery of the Miura transformation opened the modern soliton theory [GGKM]. Using this, the inverse scattering method was discovered and developed [L]. In such developments of the theory, there appeared several algebraic solutions of integrable nonlinear differential equations in the 1970s [Kr] (references in [BBEIM, MT, TD]). These solutions were obtained by arguments on the behaviours of algebraic or meromorphic functions around infinite points of corresponding algebraic curves. Their arguments are based on certain existence theorems of holomorphic functions and the theorem of identity of analytic functions. They played very important roles and contributed to the developments of the soliton theories in the 1980s. They are especially connected to the abstract theories of modern mathematics, which are, now, called infinite-dimensional analysis [DKJM, SN, SS]. Thus, needless to say, they also have important meanings.

However their solutions are not, sometimes, enough to have effects on some concrete computations of algebraic solutions, for example the cases in which one wishes to determine

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coefficients of higher degrees of Laurent or Taylor expansions of the solutions and in which one wishes to draw a graph of solutions without computation of transcendental equations. Recently theories of hyperelliptic functions which were discovered in the 19th century [Ba1, Ba2, Ba3, Kl1] have been re-evaluated in various fields [BEL1, BEL2, BEL3, CEEK, EE, EEK, EEL, EEP, G, Ma2, Ma3, Ma4, Ma5, N, Ô1, Ô2]. One purpose of the re-evaluations seems to be to reconstruct and to refine the modern soliton theory and algebraic geometry.

The theories of the 19th century are very concrete. First they chose a hyperelliptic curve. After fixing the curve, they considered the meromorphic functions over it. In the 19th century, the study of the hyperelliptic functions was very important as a generalization of elliptic functions, and was performed by Weierstrass, Klein, Jacobi and so on [D, Kl2]. Surprisingly, even in the 19th century, there appeared most of the tools and objects in soliton theories. In fact, Baker found the Korteweg–de Vries (KdV) hierarchy and Kadomtsev–Petviashvili (KP) equations around 1898 in terms of bilinear operators, Paffians, symmetric functions and so on [Ba3, Ma3].

As we enter the 21st century now, we can look around both of them. It is very interesting to compare the theory of soliton equations at the end of the last century with that of hyperelliptic functions in the 19th century. Both have merits and demerits. When one wishes for concrete expressions, the theories of the 19th century are very convenient. However, if one wishes to use the framework of infinite-dimensional analysis, the older theories might be useless.

This paper is on the hyperelliptic solutions of the modified KdV equation (MKdV) related to hyperelliptic curves with higher genus by employing the fashion in the 19th century. In the previous report [Ma2], I constructed those of genus one and two associated with loop solitons and the statistical mechanics of elastic rods. As mentioned in [Ma1], when one considers the statistical mechanics (or quantization) of ideal thin elastic rods in a plane, for example ideal polymers in a plane, the MKdV equation naturally appears and algebraic solutions (hyperelliptic solutions, or quasi-periodic solutions) of the MKdV equation have physical meaning. It is worthwhile to note that it is not trivial that the algebraic solution with higher genus is related to a physical system in low-energy physics; even for many cases which are known as physical models of solitons, quasi-periodic solutions with higher genus are often out of the range of the approximations on the models. Thus it should be noted that our investigation of the algebraic solutions of the MKdV equation is directly related to the physical system of statistical mechanics of polymers in a plane [Ma1, Z].

This paper is one of two reports as the extensions of [Ma2] to arbitrary genus. In another one [Ma5] we give a geometrical investigation of the loop solitons using the result of this paper, whereas in this paper we shall concentrate our attention on the analytic arguments on the algebraic solutions of the MKdV equations. In this paper, using only the Weierstrass al-function [W], [Ba2, p 340], we shall construct the hyperelliptic solutions of the MKdV equation in theorem 3.2. In the construction, we shall not essentially use any θ-functions or Baker–Akhiezer functions though the algebro-geometric solutions in terms of Baker–Akhiezer functions were obtained by Gesztesy and Holden recently [GH1]; they did not mention a connection between their solutions and the al-function. Our solutions are explicitly constructed without ambiguous parameters for arbitrary hyperelliptic curves and our construction might be, I believe, simpler than that in terms of Baker–Akhiezer functions. In the construction, we translate the differential operators in the Jacobian into those in the curve itself and evaluate the related terms by using residual computations and symmetries. As the Weierstrass al-function should be regarded as a generalization of the Jacobi elliptic functions and in a calculus of Jacobi elliptic functions, we do not usually need θ-functions; our construction is more natural.

In the construction of the hyperelliptic solutions of the MKdV equation, we compare the hyperelliptic $\wp$ and al-functions. As the hyperelliptic $\wp$ functions are connected with the finite-type solutions of the KdV equation [BEL1, BEL2, Ma3], we can investigate the Miura transformation between finite-type solutions of the KdV equations and MKdV equations. In remark 3.3 and section 4, we show that the Miura transformation for the finite-type solutions is also natural from several viewpoints. Here we should remark that the comparison is essentially the same as Baker's arguments in [Ba3]; there he found the KdV hierarchy and KP equation as relations among the $\wp$ functions [Ba3, Ma3]. Further, we note that, even though they did not mention it, Eilbeck, Enolskii and Kostov also found a similar relation between $\wp$ and al in their computations of the vector nonlinear Schrödinger equation [EEK].

Due to [GGKM], any solution of the KdV equation can be regarded as a potential in a one-dimensional Schrödinger equation whose spectrum is preserved for the time development of the solutions. From the spectral theory [BBEIM, MT], the spectrum is determined by a characteristic relation over the complex number, which is identified with a 'hyperelliptic curve' with a certain genus including infinite genus. For the finite case, the solutions of the KdV equation are given as finite-type solutions associated with the hyperelliptic curve. However in general the genus of the characteristic relation is not finite. Even if the characteristic relation has infinite genus, we can essentially deal with the solution of the KdV equation as in the case of finite genus by taking the limit of the genus to infinity; though there are some obstacles, we can overcome them [MT]. Thus we could recognize that our remarks in sections 3 and 4 for the finite-genus case exhibit essentials of the Miura transformation. Though Gesztesy and Holden [GH2] also considered the Miura transformation from the viewpoint of algebro-geometrical computations of the MKdV equation, our viewpoint on the remarks differs from theirs. Our remarks give another aspect of the Miura transformation.

2. Differentials of a hyperelliptic curve

In this section, we shall give the conventions which express the hyperelliptic functions in this paper. As there is a good self-contained paper on theories of hyperelliptic sigma functions [BEL2] besides [Ba1, Ba2, Ba3], we shall give them without explanations and proofs.

We denote the set of complex numbers by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

Convention 2.1. *We deal with a hyperelliptic curve C_g of genus g ($g > 0$) given by the affine equation*

$$y^2 = f(x) = \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0 = P(x)Q(x), \tag{2.1}$$

where $\lambda_{2g+1} \equiv 1$ and the λ_j are complex numbers. We use the expressions

$$\begin{aligned} f(x) &= (x - b_1)(x - b_2)\cdots(x - b_{2g})(x - b_{2g+1}),\\ Q(x) &= (x - c_1)(x - c_2)\cdots(x - c_g)(x - c),\\ P(x) &= (x - a_1)(x - a_2)\cdots(x - a_g), \end{aligned} \tag{2.2}$$

where the a_j, b_j, c_j and c are complex values and $b_{2g+2} = \infty$.

Proposition 2.2. *There exist several symmetries which express the same curve C_g.*

(1) Translational symmetry: for $\alpha_0 \in \mathbb{C}$, $(x, y) \to (x + \alpha_0, y)$, with $b_j \to b_j + \alpha_0$.
(2) Dilatation symmetry: for $\alpha_1 \in \mathbb{C}$, $(x, y) \to (\alpha_0 x, \alpha_0^{2g+1} y)$, with $b_j \to \alpha_0 b_j$.
(3) Inversion symmetry: for fixing b_i, $(x, y) \to (1/(x - b_i), y \prod_{j\neq i} \sqrt{b_i - b_j}/(x - b_i)^{(2g+1)/2})$ with $b_j \to 1/(b_j - b_i)$.

Definition 2.3 (see [Ba1, Ba2, BEL2, Ô1]). *For a point $(x_i, y_i) \in C_g$, we define the following quantities:*

(1) Let us denote the homology of a hyperelliptic curve C_g by

$$\mathrm{H}_1(C_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j, \tag{2.3}$$

where these intersections are given as $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.

(2) The un-normalized differentials of the first kind are defined by

$$\mathrm{d}u_1^{(i)} := \frac{\mathrm{d}x_i}{2y_i}, \quad \mathrm{d}u_2^{(i)} := \frac{x_i\mathrm{d}x_i}{2y_i}, \ldots, \quad \mathrm{d}u_g^{(i)} := \frac{x_i^{g-1}\mathrm{d}x_i}{2y_i}. \tag{2.4}$$

(3) The un-normalized complete hyperelliptic integrals of the first kind are defined by

$$\boldsymbol{\omega}' := \left[\left(\int_{\alpha_j} \mathrm{d}u_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\omega}'' := \left[\left(\int_{\beta_j} \mathrm{d}u_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\omega} := \begin{bmatrix} \boldsymbol{\omega}' \\ \boldsymbol{\omega}'' \end{bmatrix}. \tag{2.5}$$

(4) The normalized complete hyperelliptic integrals of the first kind are given by

$$\boldsymbol{\tau} := \boldsymbol{\omega}'^{-1}\boldsymbol{\omega}'', \qquad \hat{\boldsymbol{\omega}} := \begin{bmatrix} 1_g \\ \boldsymbol{\tau} \end{bmatrix}. \tag{2.6}$$

(5) The un-normalized differentials of the second kind are defined by

$$\begin{aligned} \mathrm{d}\tilde{u}_1^{(i)} &:= \frac{x_i^{g}\mathrm{d}x_i}{2y_i}, \quad \mathrm{d}\tilde{u}_2^{(i)} := \frac{x_i^{g+1}\mathrm{d}x_i}{2y_i}, \ldots, \quad \mathrm{d}\tilde{u}_g^{(i)} := \frac{x_i^{2g-1}\mathrm{d}x_i}{2y_i}, \\ &\text{and} \quad \mathrm{d}\boldsymbol{r}^{(i)} := (\mathrm{d}r_1^{(i)}, \mathrm{d}r_2^{(i)}, \ldots, \mathrm{d}r_g^{(i)}), \end{aligned} \tag{2.7}$$

$$(\mathrm{d}\boldsymbol{r}^{(i)}) := \Lambda \begin{pmatrix} \mathrm{d}\boldsymbol{u}^{(i)} \\ \mathrm{d}\tilde{\boldsymbol{u}}^{(i)} \end{pmatrix}, \tag{2.8}$$

where Λ is a $2g \times g$ matrix defined by

$$\Lambda = \begin{pmatrix} 0 & \lambda_3 & 2\lambda_4 & 3\lambda_5 & \cdots & (g-1)\lambda_{g+1} & g\lambda_{g+2} & (g+1)\lambda_{g+3} & \cdots & (2g-3)\lambda_{2g-1} & (2g-2)\lambda_{2g} & (2g-1)\lambda_{2g+1} \\ & 0 & \lambda_5 & 2\lambda_6 & \cdots & (g-2)\lambda_{g+2} & (g-1)\lambda_{g+3} & g\lambda_{g+4} & \cdots & (2g-4)\lambda_{2g} & (2g-3)\lambda_{2g+1} & 0 \\ & & 0 & \lambda_7 & \cdots & (g-3)\lambda_{g+3} & (g-2)\lambda_{g+4} & (g-1)\lambda_{g+5} & \cdots & (2g-5)\lambda_{2g+1} & 0 & \\ & & & & \ddots & \vdots & \vdots & \vdots & \cdots & 0 & & \\ & & 0 & & & \lambda_{2g-2} & 2\lambda_{2g-1} & 3\lambda_{2g+1} & & & & \\ & & & & & 0 & \lambda_{2g+1} & 0 & & & & \\ & & & & & & & & & & 0 & \end{pmatrix}. \tag{2.9}$$

(6) The complete hyperelliptic integral matrices of the second kind are defined by

$$\boldsymbol{\eta}' := \left[\left(\int_{\alpha_j} \mathrm{d}r_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\eta}'' := \left[\left(\int_{\beta_j} \mathrm{d}r_i^{(a)}\right)_{ij}\right]. \tag{2.10}$$

(7) By defining the Abel map for the gth symmetric product of the curve C_g,

$$\begin{aligned} &\boldsymbol{u} \equiv (u_1, \ldots, u_g) : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g, \\ &\left(u_k((x_1, y_1), \ldots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \mathrm{d}u_k^{(i)}\right), \end{aligned} \tag{2.11}$$

the Jacobi varieties $\mathcal{J}_g$ are defined as a complex torus,

$$\mathcal{J}_g := \mathbb{C}^g / \boldsymbol{\Lambda}. \tag{2.12}$$

Here $\boldsymbol{\Lambda}$ is a g-dimensional lattice generated by $\boldsymbol{\omega}$.

Definition 2.4. *The coordinate in $\mathbb{C}^g$ for points $(x_i, y_i)_{i=1,\ldots,g}$ of the curve $y^2 = f(x)$ is given by*

$$u_j := \sum_{i=1}^{g} \int_{\infty}^{(x_i,y_i)} \mathrm{d}u_j^{(i)}. \tag{2.13}$$

(1) The hyperelliptic σ function, which is a holomorphic function over $u \in \mathbb{C}^g$, is defined by [Ba2, p 336, p 350], [Kl1, BEL2]

$$\sigma(u) = \sigma(u; C_g) := \gamma \exp(-\tfrac{1}{2}\, {}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} u) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (\boldsymbol{\omega}'^{-1} u; \boldsymbol{\tau}), \tag{2.14}$$

where γ is a certain constant factor,

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z; \boldsymbol{\tau}) := \sum_{n \in \mathbb{Z}^g} \exp[2\pi\sqrt{-1}\{\tfrac{1}{2}\, {}^t(n+a)\boldsymbol{\tau}(n+a) + {}^t(n+a)(z+b)\}] \tag{2.15}$$

for g-dimensional vectors a and b and

$$\delta' := {}^t\begin{bmatrix} \frac{g}{2} & \frac{g-1}{2} & \cdots & \frac{1}{2} \end{bmatrix}, \qquad \delta'' := {}^t\begin{bmatrix} \frac{1}{2} \cdots \frac{1}{2} \end{bmatrix}. \tag{2.16}$$

(2) The hyperelliptic $\wp$-function is defined by [Ba2, p 370], [Ba1, BEL2]

$$\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u). \tag{2.17}$$

(3) The hyperelliptic al-function is defined by [Ba2, p 340], [W]

$$\mathrm{al}_r(u) := \gamma' \sqrt{F(b_r)}, \tag{2.18}$$

where γ' is a certain constant factor,

$$F(x) := (x - x_1) \cdots (x - x_g) = \gamma_g x^g + \gamma_{g-1} x^{g-1} + \cdots + \gamma_0, \tag{2.19}$$

$\gamma_g \equiv 1$ and γ_i are elementary symmetric functions of x_i.

Here we note that in [Ba2] Baker used another notation cl_r for a point $b_r \in \{a_j\}$ of (2.2) instead of al_r, but I shall follow the original definition of Weierstrass [W].

Proposition 2.5.

(1) $\wp_{gi}$ $(i = 1, \ldots, g)$ is the elementary symmetric functions of $\{x_1, x_2, \ldots, x_g\}$, i.e. for $(x_1, \ldots, x_g) \in \mathrm{Sym}(C^g)$ [Ba1, BEL2],

$$F(x) = x^g - \sum_{i=1}^{g} \wp_{gi} x^{i-1}. \tag{2.20}$$

(2) $U := (2\wp_{gg} + \lambda_{2g}/6)$ obeys the KdV equations [BEL2, p 52–53],

$$4\frac{\partial U}{\partial u_{g-1}} + 6U\frac{\partial U}{\partial u_g} + \frac{\partial^3 U}{\partial u_g} = 0. \tag{2.21}$$

(3) Introducing the half-period $\omega_r := \int_{\infty}^{b_r} \mathrm{d}u^{(a)}$, we have

$$\mathrm{al}_r(u) = \gamma_r'' \frac{\exp(-u\boldsymbol{\eta}'\boldsymbol{\omega}'^{-1}\omega_r)\sigma(u + \omega_r)}{\sigma(u)}, \tag{2.22}$$

where γ_r'' is a certain constant [Ba2, p 340].

Definition 2.6.

(1) A polynomial associated with $F(x)$ is defined by

$$\pi_i(x) := \frac{F(x)}{x - x_i} = \chi_{i,g-1}x^{g-1} + \chi_{i,g-2}x^{g-2} + \cdots + \chi_{i,1}x + \chi_{i,0}, \tag{2.23}$$

where $\chi_{i,g-1} \equiv 1$, $\chi_{i,g-2} = (x_1 + \cdots + x_g) - x_i$, and so on.

(2) We shall introduce $g \times g$ matrices,

$$W := \begin{pmatrix} \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,g-1} \\ \chi_{2,0} & \chi_{2,1} & \cdots & \chi_{2,g-1} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{g,0} & \chi_{g,1} & \cdots & \chi_{g,g-1} \end{pmatrix}, \qquad \mathcal{Y} := \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_g \end{pmatrix},$$
$$\mathcal{F}' := \begin{pmatrix} F'(x_1) & & & \\ & F'(x_2) & & \\ & & \ddots & \\ & & & F'(x_g) \end{pmatrix}, \tag{2.24}$$

where $F'(x) := \mathrm{d}F(x)/\mathrm{d}x$.

Lemma 2.7.

(1) The inverse matrix of W is given by $W^{-1} = \mathcal{F}^{-1}V$, where V is the Vandermond matrix,

$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_g \\ x_1^2 & x_2^2 & \cdots & x_g^2 \\ \vdots & \vdots & & \vdots \\ x_1^{g-1} & x_2^{g-1} & \cdots & x_g^{g-1} \end{pmatrix}. \tag{2.25}$$

(2) By letting $\partial_{u_i} := \partial/\partial u_i$ and $\partial_{x_i} := \partial/\partial x_i$, we obtain

$$\begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_g} \end{pmatrix} = 2\mathcal{Y}\mathcal{F}'^{-1}\, {}^tW \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_g} \end{pmatrix}. \tag{2.26}$$

Proof. (1) is obvious by direct substitution and uniqueness of an inverse matrix. We should pay attention to the fixed parameters in the partial differential in (2). $\frac{\partial}{\partial u_i}$ means $\left(\frac{\partial}{\partial u_i}\right)_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_g}$ where indices are fixed parameters. Since

$$\mathrm{d}x_i = \sum_{j=1}^{g} \left(\frac{\partial x_i}{\partial u_j}\right)_{u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_g} \mathrm{d}u_j, \tag{2.27}$$

we have

$$\left(\frac{\partial}{\partial u_i}\right)_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_g} = \sum_{j=1}^{g} \left(\frac{\partial x_j}{\partial u_i}\right)_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_g} \frac{\partial}{\partial x_i}. \tag{2.28}$$

Comparing (2.27) with the definitions (2.4), (2.11) and (2.13), (2) is proved. □

3. Hyperelliptic solutions of modified Korteweg–de Vries equation

In this section, we shall give hyperelliptic solutions of the MKdV equation and Miura transformation in terms of the theory of hyperelliptic functions developed in the 19th century.

First we shall define the MKdV equation.

Definition 3.1. *The MKdV equation is given by the form*

$$\frac{\partial}{\partial t}q + 6q^2\frac{\partial}{\partial s}q + \frac{\partial^3}{\partial s^3}q = 0, \tag{3.1}$$

where t and s are real or complex numbers.

Then we shall give our main theorem of this paper as follows.

Theorem 3.2.

(1) By letting

$$\mu^{(r)} = \frac{1}{2}\frac{\partial}{\partial u_g}\phi^{(r)}, \qquad \phi^{(r)}(u) := \frac{1}{\sqrt{-1}}\log F(b_r), \tag{3.2}$$

$\mu^{(r)}$ obeys the modified KdV equation,

$$\left(\frac{\partial}{\partial u_{g-1}} - (\lambda_{2g} + b_r)\frac{\partial}{\partial u_g}\right)\mu^{(r)} - 6\mu^{(r)2}\frac{\partial}{\partial u_g}\mu^{(r)} + \frac{\partial^3}{\partial u_g^3}\mu^{(r)} = 0. \tag{3.3}$$

(2) We have the Miura relation,

$$2\wp_{gg} + \lambda_{2g} + b_r = \mu^{(r)2} + \sqrt{-1}\frac{\partial}{\partial u_g}\mu^{(r)}. \tag{3.4}$$

Before proving the theorem, we shall remark on the meanings of our theorem.

Remark 3.3.

(1) For arbitrary hyperelliptic curves, we can construct $\mu^{(r)}$ in (3.2) which obeys the MKdV equation in terms of the hyperelliptic al-function, $\mathrm{al}_r = \gamma'\sqrt{F(b_r)}$. This means that we present all finite-type solutions of the MKdV equation which are expressed by meromorphic functions over hyperelliptic curves up to the symmetries of proposition 2.2. Due to (2.14) and (2.22), our solution is connected with that in [GH1].

(2) Using the Miura transformation, the theorem and our proof mean that we have another proof of proposition 2.5 (2) on the hyperelliptic solutions of the KdV equation beside [BBEIM, BEL1, BEL2].

(3) There is a map from $\mathrm{Sym}^g(C_g)$ to $\mathbb{C}P^1$

$$\begin{array}{ccc} \{(x_1, x_2, \ldots, x_g)\} & \xrightarrow{\ u\ } & \mathcal{J} \\ & & \big\downarrow \wp_{gg} \\ & & \{x_1 + \cdots + x_g\} \end{array} \tag{3.5}$$

and this characterizes the $\wp_{gg}$ function [BBEIM]. It is noted that we can regard this map as a global affine map between the affine spaces, $\mathrm{Sym}^g(C_g)$ and $\mathbb{C}$. The global translation and dilation $x_i \to ax_i + b$ for all i $(a, b \in \mathbb{C})$ makes $\wp_{gg} \to a\wp_{gg} + gb$ but the curve C_g does not change due to proposition 2.2. This translation freedom should be one of the keys of the inverse scattering method [GGKM]. Further Miura transformation connects between objects which are invariant and variant for translation respectively. In fact for the translation, Weierstrass al-functions are invariant. This reminds us of a stabilizer in homogeneous space.

Further we remark that in the property of an infinite point, $\infty + a = \infty$, the invariance can be translated to an analysis at an infinite point. With the properties of the degenerate curve $y^2 = P(x)^2 x$ as mentioned in [Ma4], Miura transformation can be interpreted as Bäcklund transformation in a certain case.

(4) From the point of view of study of symmetric functions, $F(x)$ is a generator of the elementary symmetric functions, whereas from (3.9) the definition of $\mu^{(r)}$ in (3.2) resembles the generator of the power sum symmetric function [Mac]

$$\frac{\mathrm{d}}{\mathrm{d}x}\log F(x) = \frac{1}{x}\sum_{j=0}^{\infty}\left[\sum_{i=1}^{n} x_i^j\right]\frac{1}{x^j}. \tag{3.6}$$

(5) Due to the Miura transformation, the differential $\partial\mu^{(r)}/\partial u_g$ can be expressed by a polynomial of $\wp_{gg}$ and $\mu^{(r)}$. In other words, it connects categories of analysis and algebra. In fact, the Miura transformation can be translated into a language in the theory of a geometrical Dirac equation of a curve in a complex plane, or the Frenet–Serret relation [Ma5]. There the Miura transformation is interpreted as an integrability condition in the $\mathcal{D}$-module (see [Bj, p 12–13], [Ma5]).

(6) In [EEK], Eilbeck *et al* studied the vector nonlinear Schrödinger equations and obtained a formula ((3.8) in [EEK]) which is essentially the same as (3.4), even though they did not mention a relation between their formula and the Miura transformation.

Here we shall mention our plan to prove the theorem. First we shall prove the Miura relation (3.4). As the first step in its proof, we shall translate the operation of $\partial/\partial u_g$ into that of $\partial/\partial x_i$ and compute (3.2). Secondly we shall evaluate the obtained terms by using the symmetries of summation and residual computation over the curve. Then we obtain the Miura relation (3.4). Next, we shall investigate the right-hand side in an integration version of (3.3). If we show that $\phi^{(r)}$ in (3.2) obeys

$$4\left(\frac{\partial}{\partial u_{g-1}} - (\lambda_{2g} + b_r)\frac{\partial}{\partial u_g}\right)\phi^{(r)} + \frac{1}{2}\left(\frac{\partial}{\partial u_g}\phi^{(r)}\right)^3 + \frac{\partial^3}{\partial u_g^3}\phi^{(r)} = 0, \tag{3.7}$$

we have a solution of (3.3) by differentiating (3.7) in u_g again. In the derivation, we shall use the Miura relation and correspondences between $\partial/\partial u_{g-1}$ and the $\partial/\partial x_i$. We shall obtain (3.7) at (3.31).

Now let us prove the theorem. From (2.26), we shall express the u by the affine coordinates x_i,

$$\frac{\partial}{\partial u_g} = \sum_{i=1}^{g}\frac{2y_i}{F'(x_i)}\frac{\partial}{\partial x_i}, \qquad \frac{\partial}{\partial u_{g-1}} = \sum_{i=1}^{g}\frac{2y_i\chi_{i,g-1}}{F'(x_i)}\frac{\partial}{\partial x_i}. \tag{3.8}$$

Hence we have

$$\mu^{(r)} \equiv \frac{1}{2\sqrt{-1}}\frac{\partial}{\partial u_g}\log F(b_r) = \frac{1}{\sqrt{-1}}\sum_{i=1}^{g}\frac{y_i}{F'(x_i)(x_i - b_r)} = \frac{1}{\sqrt{-1}}\sum_{j=1}^{\infty}\sum_{i=1}^{g}\frac{y_i}{F'(x_i)b_r}\frac{x_i^j}{b_r^j}, \tag{3.9}$$

$$\frac{\partial}{\partial u_{g-1}}\log F(b_r) = \sum_{i=1}^{g}\frac{2y_i\chi_{i,g-1}}{F'(x_i)(x_i - b_r)}. \tag{3.10}$$

Roughly speaking, we wish to compute

$$\frac{1}{2}\frac{\partial^2}{\partial u_g^2}\log F(b_r) = \sum_{j=1,i=1}^{g}\frac{y_j}{F'(x_j)}\frac{\partial}{\partial x_j}\frac{2y_i}{F'(x_i)(x_i - b_r)}, \tag{3.11}$$

and express it by a lower derivative.

Here we shall summarize the derivatives of $F(x)$, which are shown by direct computations.

Lemma 3.4.

$$(1) \qquad \frac{\partial}{\partial x_i} F(x) = -\frac{F(x)}{x - x_j}. \tag{3.12}$$

$$(2) \qquad \left[\frac{\partial}{\partial x} F(x)\right]_{x=x_i} = \prod_{j\neq i}(x_i - x_j). \tag{3.13}$$

$$(3) \qquad \frac{\partial}{\partial x_k} F'(x_k) = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} F(x)\right]_{x=x_k}. \tag{3.14}$$

In order to evaluate (3.11), we shall first show the next lemma.

Lemma 3.5.

$$(1) \qquad \sum_{k=1}^{g} \frac{1}{F'(x_k)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-b_r)F'(x)}\right)\right]_{x=x_k} = \lambda_{2g} + b_r + 2\wp_{gg}. \tag{3.15}$$

$$(2) \qquad \left(\sum_k \frac{y_k}{(x-x_k)F'(x_k)}\right)^2 = \sum_{k,l,k\neq l} \frac{2y_k y_l}{(x-x_k)(x_k-x_l)F'(x_k)F'(x_l)} + \sum_k \frac{y_k^2}{(x-x_k)^2F'(x_k)^2}. \tag{3.16}$$

Proof. In order to prove (1), we shall consider an integral over a contour in C_g and follow four steps as follows.

(1) Let ∂C_g^o be a boundary of a polygon representation C_g^o of the curve C_g,

$$\oint_{\partial C_g^o} \frac{f(x)}{(x-b_r)F(x)^2}\,\mathrm{d}x = 0. \tag{3.17}$$

(2) The divisor of the integrand of (1) is

$$\left(\frac{f(x)}{(x-b_r)F(x)^2}\,\mathrm{d}x\right) = \sum_{i=1,b_i\neq b_r}^{2g+1}(b_i,0) - 2\sum_{i=1}^{g}(x_i,y_i) - 2\sum_{i=1}^{g}(x_i,-y_i) - 3\infty. \tag{3.18}$$

(3) Noting that the local parameter t at ∞ is $x = 1/t^2$,

$$\mathrm{res}_\infty \frac{f(x)}{(x-b_r)F(x)^2}\,\mathrm{d}x = -2(\lambda_{2g} + b_r + 2\wp_{gg}). \tag{3.19}$$

(4) Noting that the local parameter t at $(x_k, \pm y_k)$ is $t = x - x_k$,

$$\mathrm{res}_{(x_k,\pm y_k)} \frac{f(x)}{(x-b_r)F(x)^2}\,\mathrm{d}x = \frac{1}{F'(x_k)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-b_r)F'(x)}\right)\right]_{x=x_k}. \tag{3.20}$$

By arranging these, we obtain (3.15). On the other hand, (2) can be proved by using a trick: for $i \neq j$,

$$\frac{1}{(x_j-x_i)(x_i-b_r)} + \frac{1}{(x_i-x_j)(x_j-b_r)} = \frac{1}{(x_j-b_r)(x_i-b_r)}. \tag{3.21}$$

Then we have (3.16). □

Let us compute (3.11), which is $\sqrt{-1}\partial\mu^{(r)}/\partial u_g$, as follows:

$$\frac{1}{2}\frac{\partial^2}{\partial u_g^2}\log F(b_r) = 2\sum_{i=1}^{g}\frac{y_i}{F'(x_i)}\frac{\partial}{\partial x_i}\frac{y_i}{F'(x_i)(x_i-b_r)} + 2\sum_{i=1,j=1,j\neq i}^{g}\frac{y_j}{F'(x_j)}\frac{\partial}{\partial x_j}\frac{y_i}{F'(x_i)(x_i-b_r)}. \tag{3.22}$$

The first term in the right-hand side of (3.22) becomes

$$\sum_{i=1}^{g}\frac{y_i}{F'(x_i)}\left(\frac{1}{y_iF'(x_i)(x_i-b_r)}\frac{\mathrm{d}f(x_i)}{\mathrm{d}x_i} - \frac{y_i}{F'(x_i)(x_i-b_r)}\left[\frac{\partial}{\partial x}F(x)\right]_{x=x_i} - \frac{2y_i}{F'(x_i)(x_i-b_r)^2}\right). \tag{3.23}$$

Using the lemma 3.4 (1), it can be modified to

$$\sum_{i=1,j=1,j\neq i}^{g}\frac{1}{F'(x_k)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-b_r)F'(x)}\right)\right]_{x=x_i} - \sum_{i=1}^{g}\frac{f(x_i)}{F'(x_i)^2}\left(\frac{1}{(x_i-b_r)^2}\right). \tag{3.24}$$

Using (3.15), the first term in (3.24) is expressed well. On the other hand, the second term in the right-hand side of (3.22) is computed to

$$2\sum_{i=1,j=1,j\neq i}^{g}\frac{y_iy_j}{F'(x_j)F'(x_i)}\frac{1}{(x_j-x_i)(x_i-b_r)}. \tag{3.25}$$

From lemma 3.5 (2), the second term in (3.24) and (3.25) becomes

$$\frac{1}{4}\left(\sum_{i=1}^{g}\frac{2y_i}{F'(x_i)}\frac{1}{(x_i-b_r)}\right)^2 \equiv -\mu^{(r)2}. \tag{3.26}$$

Hence we have theorem 3.2 (2).

Next we shall prove (3.3) as follows: from theorem 3.2 (2) ($\gamma_{g-1} \equiv -\wp_{gg}$ due to (2.19) and (2.20)) we have

$$\frac{\partial}{\partial u_g}\mu^{(r)} = \frac{1}{\sqrt{-1}}\left(-2\gamma_{g-1} + \lambda_{2g} + b_r - \mu^{(r)2}\right). \tag{3.27}$$

We perform the differential in u_g on both sides again,

$$\begin{aligned}\frac{\partial^2}{\partial u_g^2}\mu^{(r)} &= \frac{1}{\sqrt{-1}}\frac{\partial}{\partial u_g}\left(-2\gamma_{g-1} + \lambda_{2g} + b_r - \mu^{(r)2}\right)\\ &= \frac{1}{\sqrt{-1}}\left(-2\sum_{i=1}^{g}\frac{2y_i}{F'(x_i)} - 2\mu^{(r)}\frac{\partial}{\partial u_g}\mu^{(r)}\right)\\ &= \frac{-2}{\sqrt{-1}}\left(\sum_{i=1}^{g}\frac{2y_i}{F'(x_i)} + \frac{1}{\sqrt{-1}}\mu^{(r)}\left[-2\gamma_{g-1} + \lambda_{2g} + b_r - \mu^{(r)2}\right]\right).\end{aligned} \tag{3.28}$$

In the second step, we used (3.27) again and

$$\frac{\partial}{\partial u_g}\gamma_{g-1} = -\sum_{i=1}^{g}\frac{2y_i}{F'(x_i)}. \tag{3.29}$$

Noting $\chi_{i,g-1} = \gamma_{g-1} - x_i$, the first and second terms in the parenthesis are

$$\sum_{i=1}^{g} \frac{2y_i}{F'(x_i)} - \frac{2}{\sqrt{-1}}\gamma_{g-1}\mu^{(r)} = \sum_{i=1}^{g} \frac{2y_i}{F'(x_i)}\frac{x_i - b_r}{x_i - b_r} + \gamma_{g-1}\sum_{i=1}^{g} \frac{2y_i}{F'(x_i)(x_i - b_r)}$$

$$= \frac{\partial}{\partial u_{g-1}} \log F(b_r) - \frac{2}{\sqrt{-1}} b_r \mu^{(r)}. \tag{3.30}$$

Thus we obtain (3.7),

$$-\frac{1}{4}\frac{\partial^3}{\partial u_g^3}\phi^{(r)} = \left(\frac{\partial}{\partial u_{g-1}} - (\lambda_{2g} + b_r)\frac{\partial}{\partial u_g}\right)\phi^{(r)} + \frac{1}{8}\left(\frac{\partial}{\partial u_g}\phi^{(r)}\right)^3. \tag{3.31}$$

We differentiate (3.31) in u_g again and then obtain (3.3). Hence we completely prove our theorem 3.2.

4. Discussion

We obtained the hyperelliptic solutions of the MKdV equation in terms of hyperelliptic al-functions without any theta-functions, which is in contrast to the approach of Gesztesy and Holden [GH1] and others ([BBEIM] and references therein). For an arbitrary hyperelliptic curve, we can write down its explicit function form by the al-function. Miura transformation means a connection between the $\wp_{gg}$-functions and the $\mu^{(r)}$ in (3.2) which are differentials of the hyperelliptic al-functions; this aspect also differs from the study of the Miura transformation by Gesztesy and Holden [GH2].

According to [Kl2] and [D], Weierstrass found the σ function for an elliptic curve, which he called the Al-function in honour of Abel, and his approach was to investigate divisor decomposition of a function,

$$y = 4y_1y_2y_3, \tag{4.1}$$

for an elliptic curve $y^2 = 4x^3 + g_3x + g_4 = 4(x-e_1)(x-e_2)(x-e_3)$; $y_i = \sqrt{x - e_i}$. These y_i differ from his σ-function itself but correspond to our al-function (2.18). y_i is expressed by a ratio of two σs. The functions y_i are generators of Jacobi sn, cn and dn functions. Comparison of the y_i with $\wp$ gives us fruitful information for the elliptic curve case. It is useful to identify these *sn* with the y_i.

For the case of hyperelliptic curves, Weierstrass called the y_i themselves al-functions and defined them as in (2.18) with explicit constant factor γ' [D, Kl2, W]. (The hyperelliptic sigma function is also a factor of the al_r as in (2.22), which was discovered by Klein [Kl1].) In [Ba3], Baker found the KdV hierarchy and KP equation around 1898 by comparing $F(x)$ and $\wp$. It is also important that we roughly identify with $F(b_r)$ and al_r and recognize that $\mu^{(r)}$ originates from $F(b_r)$. Miura transformation is a key to the relations between $F(b_r)$ and $\wp_{gg}$ as shown in this paper. In fact, the techniques in the proof of theorem 3.2 are essentially contained in [Ba3]. In other words we should regard a 'Miura transformation' as existing behind discoveries of the KdV hierarchy and KP equation by Baker, from [Ba3].

In the theory of elliptic functions, we know, by experience, that Jacobi elliptic functions rather than the $\wp$ function play important roles in physics, whereas $\wp$ is more essential in number theories and algebraic geometry. In fact, in the book [BF], there are many relations on Jacobi elliptic functions but not so many on $\wp$. Thus a question might arise of why there are such differences between Jacobi elliptic functions, special al-functions and the $\wp$ function. One of the answers is that the al-function is invariant for the affine transformation as mentioned in remark 3.3. On the other hand, $\wp$ is not. Of course for the standard representation of an elliptic curve, i.e. $y^2 = 4x^3 - g_2x + g_3$, in order to make the coefficient of x^2 vanish, the translation freedom is constrained. However we can find $\wp$ even for $y^2 = 4x^3 + g_1x^2 + g_2x + g_3$ and the nature of such a $\wp$ is x itself in the affine space. Thus al and $\wp$ exist in different categories. As we show in [Ma2] and [Ma5], the al-function is associated with the differential geometry and solutions of Dirac equations, while $\wp$ is connected to algebraic geometry; $\wp$ is the affine coordinate of the curve.

Hence we conclude that Miura transformation is a connection between the worlds of $\wp$ and the al. I think that the researchers in the 19th century might have implicitly already recognized these facts. However as we can, now, look around both theories of 'Abelian functions', in the last century and the 19th century, I think that we should unify them and develop a new theory beyond them.

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